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# Optimal Transport, Convection, Magnetic Relaxation and Generalized Boussinesq equations

*Yann Brenier\**

## Abstract

We establish a connection between Optimal Transport Theory (see [Vi] for instance) and classical Convection Theory for geophysical flows [Pe]. Our starting point is the model designed few years ago by Angenent, Haker and Tannenbaum [AHT] to solve some Optimal Transport problems. This model can be seen as a generalization of the Darcy-Boussinesq equations, which is a degenerate version of the Navier-Stokes-Boussinesq (NSB) equations.

In a unified framework, we relate different variants of the NSB equations (in particular what we call the generalized Hydrostatic-Boussinesq equations) to various models involving Optimal Transport (and the related Monge-Ampère equation [Br, Ca]). This includes the 2D semi-geostrophic equations [Ho, CNP, BB, CGP, Lo] and some fully non-linear versions of the so-called high-field limit of the Vlasov-Poisson system [NPS] and of the Keller-Segel for Chemotaxis [KS, JL, CMPS].

Mathematically speaking, we establish some existence theorems for local smooth, global smooth or global weak solutions of the different models. We also justify that the inertia terms can be rigorously neglected under appropriate scaling assumptions in the Generalized Navier-Stokes-Boussinesq

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equations.

Finally, we show how a “stringy” generalization of the AHT model can be related to the magnetic relaxation model studied by Arnold and Moffatt to obtain stationary solutions of the Euler equations with prescribed topology (see [AK, Mo, Mo2, Sc, VMI, Ni]).

## 1 The Angenent-Haker-Tannenbaum model for Optimal Transport problems

In this section, we consider the model introduced by Angenent, Haker and Tannenbaum [AHT]. This model was designed in order to seek the solutions of some optimal transport problems as equilibrium states of a suitable dynamical system that could be efficiently solved on a computer. The concrete applications have been computer vision, image registration and image warping.

### 1.1 Optimal transport and rearrangements

Let us briefly recall some typical results in Optimal Transport Theory, such as the polar factorization of maps. More precisely, let  $D$  be the closure of a bounded connected open set in  $R^d$ , with a boundary of zero  $d$ -dimensional Lebesgue measure. Up to a rescaling, we assume the Lebesgue measure of  $D$  to be 1. Given an  $L^2$  map  $y : D \rightarrow R^d$ , we call image measure of the Lebesgue measure on  $D$  by  $y$  the unique nonnegative (Borel) measure  $\mu$  defined by:

$$\int_{R^d} f(x) \mu(dx) = \int_D f(y(a)) da, \quad (1)$$

for all compactly supported continuous functions  $f$  on  $R^d$ . We have

$$\int_{R^d} \mu(dx) = 1, \quad \int_{R^d} |x|^2 \mu(dx) = \int_D |y(a)|^2 da,$$

which means that  $\mu$  belongs to the set  $Prob_2(R^d)$  of all (Borel) probability measures  $\mu$  on  $R^d$  such that  $\int |x|^2 \mu(dx) < \infty$ . In this space, we say that a sequence  $\mu_n$  converges tightly to  $\mu$  in  $Prob_2(R^d)$ , if:

$$\int_{R^d} f(x) \mu_n(dx) \rightarrow \int_{R^d} f(x) \mu(dx)$$

for all continuous function  $f$  on  $R^d$  such that

$$\sup_{x \in R^d} \frac{|f(x)|}{1 + |x|^2} < +\infty.$$

Given two  $L^2$  maps  $y$  and  $z$  from  $D$  to  $R^d$ , we say that they are rearrangement of each other if they define the same image measure. When  $y$  is a rearrangement of the identity map, we say, in short, that  $y$  is Lebesgue measure preserving.

Next, we define the class of maps with convex potential:

**Definition 1.1** *We say that an  $L^2$  map from  $D$  to  $R^d$  belongs to the class  $C$  of maps with a convex potential if there is a lower continuous convex function  $p : R^d \rightarrow ]-\infty, +\infty]$  such that, for Lebesgue almost every point  $x \in D$ ,  $p$  is differentiable at  $x$  and its gradient  $\nabla p(x)$  coincides with  $y(x)$ .*

Then, we get from [Br]:

**Theorem 1.2** *(Rearrangements with convex potentials)*

*For each  $L^2$  map  $y : D \rightarrow R^d$  there is a unique rearrangement map with a convex potential  $y^* \in C$ . The map  $y^*$  depends on  $y$  only through the associated measure  $\mu$  defined by (1).*

*In addition, the nonlinear operator  $y \in L^2 \rightarrow y^* \in L^2$  is continuous as well as the induced operator  $\mu \in \text{Prob}_2(R^d) \rightarrow y^* \in L^2$ , with respect to the tight convergence.*

We get more precise results if  $y$  is a non degenerate map, in the sense that the pre-image of every Lebesgue negligible set is also negligible:

**Theorem 1.3** *(Polar factorization of maps [Br])*

*Let  $y$  be a non degenerate  $L^2$  map from  $D$  to  $R^d$ . Then, there is a unique “polar factorization”  $y = Y \circ X$  where  $Y$  belongs to  $C$  and  $X$  is a Lebesgue measure preserving map of  $D$ . In this decomposition,  $Y$  is the unique rearrangement  $y^*$  of  $y$  in  $C$  and  $X$  is the unique measure preserving map of  $D$  that minimizes  $\int_D |X(a) - y(a)|^2 da$ . In addition,  $X$  can be written:*

$$X(a) = (\nabla \Phi)(y(a)), \quad \text{a.e. } a \in D, \quad (2)$$

*where  $\Phi$  is a convex Lipschitz function defined on  $R^d$ .*

For (much) more results on optimal transport, we refer to Villani’s textbook [Vi]. The expression “optimal transport” comes from the fact that  $y^*$ , among all possible rearrangements  $y$  of  $y^0$ , is the unique minimizer of the “transportation cost”.

$$\int_D |y(x) - x|^2 dx,$$

where  $|\cdot|$  denotes the Euclidean norm. The name “map with convex potential” is due to Caffarelli [Ca]. The concept of polar factorization has been extended to Riemannian manifolds by McCann [Mc]. Examples of concrete applications of optimal transport techniques to natural and computer sciences can be found in [FMMS, HZTA].

## 1.2 The AHT model

The AHT model is an attempt to get the unique rearrangement  $y^*$  of  $y^0$ , with convex potential, as the equilibrium state at  $t = +\infty$  of the following set of evolution equations:

$$\partial_t y + (v \cdot \nabla)y = 0, \quad (3)$$

$$Kv + \nabla p = y, \quad \nabla \cdot v = 0, \quad (4)$$

where  $y = y(t, x) \in R^d$ ,  $v = v(t, x) \in R^d$ ,  $p = p(t, x) \in R$  depend on  $t \geq 0$  and  $x \in D$ , and  $K$  is a “dissipative” operator to be chosen, for instance  $K = I$  or  $K = -\Delta$ . In these “AHT” equations, we denote the inner product in  $R^d$  by  $\cdot$  and we use notations:

$$\nabla_i = \frac{\partial}{\partial x_i}, \quad v \cdot \nabla = \sum_{j=1,d} v_j \frac{\partial}{\partial x_j}, \quad \Delta = \sum_{j=1,d} \frac{\partial^2}{\partial x_j^2}.$$

The boundary conditions for the AHT system (3,4) are:

- i) the initial value of  $y$  at  $t = 0$ ,  $y(0, x) = y^0(x)$ ,
- ii)  $v$  is parallel to the boundary  $\partial D$  if  $K = I$  and  $v = 0$  along the boundary if  $K = -\Delta$ .

Notice that neither  $p$  nor  $v$  need initial conditions. As a matter of fact, as  $K = I$ , the second AHT equation (4) just corresponds to the “Helmholz decomposition” of  $y$  as a sum of a gradient field and a divergence-free field parallel to the boundary  $\partial D$ . The field  $p$  can be recovered by solving the Poisson problem:

$$\Delta p = \nabla \cdot y,$$

inside  $D$  with inhomogeneous Neumann condition  $\nabla p \cdot n = y \cdot n$  along the boundary, where  $n$  denotes the outward normal. Then, we get:  $v = y - \nabla p$ . So, we can write:  $v = Py$ , where  $P$  is a linear singular integral operator bounded in all  $L^p$  space for  $1 < p < +\infty$ , provided that the domain  $D$  is smooth enough. In the case  $K = -\Delta$ , in a similar way we can write

$v = P_\Delta y$ , where  $P_\Delta$  is a linear singular integral operator bounded from  $L^p$  to the Sobolev space  $W^{2,p}$ , for all  $1 < p < +\infty$ ,  $D$  being assumed to be smooth. Thus we can write the AHT system (3,4) in a more abstract form:

$$\partial_t y + (P_K y \cdot \nabla) y = 0, \quad (5)$$

with  $P_K = P$  if  $K = I$  and  $P_K = P_\Delta$  if  $K = -\Delta$ .

### 1.3 Expected long time behaviour of the AHT model

Let us now explain why the AHT model is expected to solve the Optimal Transport (or rearrangement) problem, at least for a large class of data. First, we observe that equation (3) expresses, at least formally, that, at each time  $t$ ,  $y(t, \cdot)$  is a rearrangement of  $y^0$ . Indeed, for any smooth compactly supported function  $f$ , we get:

$$\begin{aligned} \frac{d}{dt} \int_D f(y(t, x)) dx &= - \int_D (\nabla f)(y) \cdot (v \cdot \nabla) y dx \\ &= - \int_D v \cdot \nabla [f(y)] dx, \end{aligned}$$

(using the chain rule) which is zero, since  $v$  is divergence free and parallel to  $\partial D$  and is therefore  $L^2$  orthogonal to any gradient field. (Notice that this calculation can be made rigorous provided that  $v$  has enough regularity. According to Ambrosio's recent improvement of the DiPerna-Lions theory on ODEs [Am, DL], it is enough that  $v$  belongs to  $L^1_{loc}(R_+, BV(D, R^d))$ .) Next, we get the following balance law for the AHT

$$\frac{d}{dt} \int_D \frac{1}{2} |y(t, x) - x|^2 dx = - \int_D (v \cdot K v)(t, x) dx, \quad (6)$$

which implies, at least formally, that  $y$  and  $v$  respectively belong to the functional spaces  $L^\infty(R_+, L^2(D, R^d))$  and  $L^2(R_+, H_K(D))$ . Here  $H_K(D)$  denotes the Hilbert space of all divergence free fields  $w(x) \in R^d$  for which  $\int_D w \cdot K w dx$  is finite with suitable boundary conditions ( $w$  parallel to  $\partial D$  as  $K = I$  or  $w = 0$  on  $\partial D$  as  $K = -\Delta$ ). The formal proof of (6) is as follows:

$$\frac{d}{dt} \int_D \frac{1}{2} |y - x|^2 dx = - \int_D (y - x) \cdot ((v \cdot \nabla) y) dx$$

(using the first AHT equation (3))

$$\begin{aligned} &= - \int_D v \cdot \nabla \left( \frac{1}{2} |y - x|^2 \right) dx - \int_D v \cdot (y - x) dx \\ &= - \int_D v \cdot (y - x) dx \end{aligned}$$

(since  $v$  is divergence free and parallel to  $\partial D$  and is therefore  $L^2$  orthogonal to any gradient field)

$$= - \int_D v \cdot (Kv + \nabla(p - \frac{1}{2}|x|^2)) dx$$

(using the second AHT equation (4))

$$= - \int_D v \cdot Kv dx.$$

At this point, we can describe the expected long time behaviour of the AHT system through the following heuristics. Since  $Kv \cdot v$  is space-time integrable, we first argue that, as  $t \rightarrow +\infty$ ,  $v$  presumably tends to zero. Then we expect  $y$  to have a definite strong limit  $y^\infty$  in  $L^2$ , which is, then, necessarily a rearrangement of  $y^0$ . Passing to the limit in the second AHT equation (4), we conclude that  $y^\infty$  must be a gradient. Therefore, at the end of the process,  $y^0$  has been rearranged as a map  $y^\infty$  with a potential. Observe that this potential needs not being convex. This is obvious in the special case when  $y^0$  is itself a map with a potential which is not convex. Indeed, then

$$y(t, x) = y^0(x), \quad v(t, x) = 0,$$

is a trivial stationary solution to the AHT equations (3,4) and we get  $y^\infty = y^0$  as a map with a non convex potential. So, we need further assumptions on  $y^0$  to be convinced that  $y^\infty$  has a chance to have a convex potential. A natural assumption is that  $y^0$  is smooth with a positive jacobian determinant valued in some interval  $[r, 1/r]$  with  $0 < r < +\infty$ . Indeed, for such an initial condition, the AHT equations have a global smooth solution  $y$  (at least in the case when  $K = -\Delta$ , as discussed in the next subsection), with a jacobian determinant that must stay in the same interval  $[r, 1/r]$ , since  $y$  is always a rearrangement of  $y^0$ . So if the convergence to  $y^\infty$ , as  $t \rightarrow +\infty$ , is strong enough, we expect  $y^\infty$  to have a convex potential and, therefore, coincide with the unique rearrangement of  $y^0$  with convex potential  $y^*$  provided by Theorem 1.2. The results obtained in [AHT] are only partial and leave as an open question this issue.

## 1.4 Wellposedness of the AHT equations

From the PDE viewpoint, it is crucial to check that the AHT system (3,4) is wellposed, which is done by Angenent, Haker and Tannenbaum in [AHT], for a class of dissipative operator  $K$  including  $K = I$ . Let us briefly discuss the wellposedness issue in the cases  $K = I$  and  $K = -\Delta$ .

For  $K = I$ , the AHT system is similar to the inviscid Burgers equation,

$$\partial_t y + (y \cdot \nabla) y = 0, \quad (7)$$

since  $P_K$  behaves like a pseudo-differential operator of order zero. Thus, the local in time existence of smooth solutions for smooth initial conditions can be obtained from rather standard energy estimates. It is a challenging and interesting open question whether the Lipschitz norm, in space, of such solutions may blow up in finite time (as it would be the case of the inviscid Burgers equation). In sharp contrast, in the case  $K = -\Delta$ , smooth solutions are clearly global in time. Indeed, from (3), we immediately get that  $|y(t, x)|$  is uniformly bounded by the sup norm of  $y^0$  that we denote by  $M^0$  and suppose, here, to be finite. Thus, because of (4), according to standard elliptic regularity theory, the  $L_t^\infty(W_x^{2,p})$  norm of  $v(t, x)$  is controlled by  $M^0$  for all finite  $p$ . Thus, the same is true for the sup norm of  $\nabla v(t, x)$ . Differentiating (3) in  $x$ , we deduce that the sup norm of  $\nabla y(t, x)$  in  $x$  cannot grow, in sup norm, faster than exponentially in  $t$  as soon as  $\nabla y^0$  has a finite sup norm. So, there is no possible blow up of the Lipschitz norm of both  $v$  and  $y$  and, therefore, by a standard argument, smooth solutions must be global in time. (Notice that the dissipative operator  $K = (-\Delta)^{1/2}$ , with appropriate boundary condition, would be borderline to get such a Lipschitz estimate. In that case  $v$  would be a priori only Log-Lipschitz, just like the Yudovich solutions of the 2D Euler equations [MP].) For more details, we refer to the paper by Angenent, Haker and Tannenbaum [AHT], where different kinds of operator  $K$  are considered.

## 1.5 Interpretation of the AHT system in terms of Convection Theory

From a Fluid Mechanics viewpoint, the AHT equations look very similar to the Boussinesq equations for convective flows, in particular to their Darcy-Boussinesq version. A classical model for Convection Theory is provided by the Navier-Stokes Boussinesq (NSB) equations that we are now going to



review with more details. Using the Boussinesq approximation, the Navier-Stokes equations for an inhomogeneous incompressible fluid subject to gravity along the  $x_d$  direction read:

$$\rho_0(\partial_t v + (v \cdot \nabla)v) - \nu \Delta v + \nabla p = y, \quad \nabla \cdot v = 0, \quad (8)$$

$$\partial_t y + (v \cdot \nabla)y = 0. \quad (9)$$

Here,  $v = v(t, x) \in R^d$  is the velocity field,  $p = p(t, x) \in R$  the pressure field,  $\rho_0 > 0$  the average density of the fluid,  $\nu$  the (constant) viscosity of the fluid, while  $y$  has only one component in the “vertical” direction  $x_d$ , which is  $-g \theta(t, x)$ , where  $g$  is the gravity constant and  $\theta(t, x)$  is the difference between the density of the fluid at  $(t, x)$  and the averaged density  $\rho_0$  of the fluid. (Usually in Convection Theory, a diffusion term is added to the advection equation for  $y$  [ID].) We recall that the Boussinesq approximation amounts to consider a variable density incompressible fluid for which the density variations are sufficiently small to be neglected in all terms except the gravity force. This approximation is widely used for ocean and atmosphere modelling [Pe]. (To the best of our knowledge the justification of this approximation is still an open problem in mathematical Fluid Dynamics, mostly because of our rather poor knowledge of the Navier-Stokes equations for inhomogeneous flows, see discussions in [Li, Ma] for instance.) By neglecting the inertia term, or equivalently by setting  $\rho_0 = 0$  in the NSB (Navier-Stokes-Boussinesq) equations, we get the simpler Stokes-Boussinesq (SB) (related to large-Prandtl-number Convection Theory as in [DOR, Wa], for instance). If, in addition, the diffusion term  $-\nu \Delta v$  is replaced by a friction term such as  $v$ , we get the Darcy-Boussinesq (DB) model. We immediately see that both the SB and the DB equations are just particular cases of the AHT model (3,4), for which the vector valued function  $y$  has only one component along the  $x_d$  axis. Indeed, the DB and SB models then respectively correspond to the choice  $K = I$  and  $K = -\Delta$  in the second AHT equation (4). According to the discussion made in subsection 1.3, we expect, for the AHT model, the  $y(t, x)$  to converge, as  $t \rightarrow +\infty$ , to a map  $y^\infty(x)$  with, hopefully, a convex potential. In the particular case of the convective DB and SB models,  $y(t, x)$  has only one component in the  $x_d$  direction, namely  $-g \theta(t, x)$ . Interestingly enough, the convergence toward a map with convex potential, *exactly* means, for the DB and SB models, that the density field tends to a density “profile”  $\rho^\infty(x_d)$ , depending only on the vertical coordinate  $x_d$ , and monotonically decreasing. This clearly corresponds, in terms of Convection Theory,

to a “stable hydrostatic equilibrium”. Notice that a similar discussion can be found in Moffatt’s paper [Mo] (section 2) as a prelude to his Magnetic Relaxation model that we will consider at the end of the present paper.

## 2 Generalized Navier-Stokes-Boussinesq equations

The interpretation of the AHT model in terms of Convection Theory suggests the following “GNSB” generalization of the NSB (Navier-Stokes-Boussinesq) equations:

$$\epsilon(\partial_t v + (v \cdot \nabla)v) + Kv + \nabla p = F(x, y), \quad \nabla \cdot v = 0, \quad (10)$$

$$\partial_t y + (v \cdot \nabla)y = G(x, y), \quad (11)$$

where  $y = y(t, x) \in R^m$  is a vector-valued function ( $m \geq 1$ , in practice  $m = d$  or  $m = 2d$  for the models discussed below),  $F$  and  $G$  are given smooth functions with bounded derivatives up to second order, respectively defined on  $R^m$  and  $R^m \times R^d$ ,  $\epsilon > 0$  is a scaling factor introduced to single out the inertia term, and  $K$  is a linear dissipative operator. Depending on the applications in view, only the following cases will be considered:  $K = 0$  (no dissipation),  $Kv = v$  (linear friction),  $Kv = -\Delta v$  (viscosity).

### 2.1 Existence theory for the GNSB equations

For simplicity, we consider in this subsection the domain  $D$  to be the unit periodic cube  $T^d = R^d/Z^d$ , in order to avoid technicalities due to spatial boundary conditions. For the three possible choices of the dissipative operator  $K$

$$Kv = 0, \quad Kv = v, \quad Kv = -\Delta v, \quad (12)$$

the existence and uniqueness of a local in time smooth solution  $(y, v)$  of the GNSB equations (10,11), for each smooth initial condition  $(y^0, v^0)$  given on the torus  $T^d$ , follow from standard theory on Euler and Navier-Stokes equations (for which we refer to [Li]).

We say that  $(y, v)$  is a weak solution if:

- 1)  $y(t, x)$  and  $v(t, x)$  depends continuously on  $t$  with values in  $L^2(D, R^d)$  (with respect to the weak topology of  $L^2$ );
- 2) For all smooth time dependent vector fields  $w(t, x)$ ,  $z(t, x)$ , with  $\nabla \cdot w = 0$ , we have:

$$\frac{d}{dt} \int v \cdot w \, dx = \int [\epsilon v \cdot (\partial_t + (v \cdot \nabla))w - Kv \cdot w + F(x, y) \cdot w] dx, \quad (13)$$

$$\frac{d}{dt} \int y \cdot z \, dx = \int y \cdot (\partial_t + (v \cdot \nabla))z + G(x, y) \cdot z \, dx, \quad (14)$$

3) The following energy inequality holds true:

$$\frac{1}{2} \frac{d}{dt} \int (\epsilon |v|^2 + |y|^2) \, dx + \int K v \cdot v \, dx \leq \int [F(x, y) \cdot v + G(x, y) \cdot y] \, dx. \quad (15)$$

When  $K = -\Delta$ , the existence of global weak solutions for the GNSB equations follows from standard arguments à la Leray combined with the DiPerna-Lions theory on ODEs [Li, DL]. They are unique in 2 space dimensions. In sharp contrast, as  $K = I$  or  $K = 0$ , nothing can be said about global weak solutions.

Concerning global smooth solutions, the existence theory is quite challenging, even in 2 space dimensions. Recently, Chae, Hou and Li [Ch, HL] have proven that Navier-Stokes Boussinesq equations (8,9) (just called Boussinesq equations in these papers) have global smooth solutions when  $d = 2$  and  $K = -\Delta$ . The same result can be readily extended to the GNSB equations (10,11), essentially because we assume the right-hand sides  $F(x, y)$  and  $G(x, y)$  of each equation to be smooth functions of  $y$  and  $v$  with bounded derivatives up to order two. (Indeed, these assumptions are enough for a straightforward adaptation of the proof of Theorem 1.1 in Chae's paper, through estimates (2.1  $\cdots$  17) in [Ch]. Some constants involved in these estimates have just to be modified to take into account the Lipschitz constants of  $F$  and  $G$ .)

So, we can summarize all these results in the following Theorem, which is nothing but a straightforward adaptation of known results:

**Theorem 2.1** *Assume the dissipative operator  $K$  to be of type (12). Then, the generalized Navier-Stokes Boussinesq equations (10,11) admit, for any smooth initial condition, a unique local smooth solution.*

*If  $K = -\Delta$ , the GNSB equations admit at least a global weak solution  $(y, v)$  (in the sense (13,14,15)) for any initial condition  $(y^0, v^0)$  in  $L^2$ . If  $d = 2$ , these weak solutions are unique. Furthermore, still for  $d = 2$ , the solutions are globally smooth for smooth initial conditions.*

## 2.2 Zero inertia limit of the GNSB equations

By zero inertia limit of the GNSB, we mean the formal limit obtained by dropping the scaling factor  $\epsilon$  in front of the inertia terms in (10,11). Namely, in Eulerian coordinates,

$$\partial_t y + (v \cdot \nabla) y = G(x, y), \quad (16)$$

$$Kv + \nabla p = F(x, y), \quad \nabla \cdot v = 0.$$

We are able to make a rigorous derivation of the zero inertia limit when  $K$  is strictly dissipative ( $K = 0$  being so excluded):

**Theorem 2.2** *Assume the dissipation operator  $K$  to be coercive in  $L^2$ , namely  $K \geq \alpha$  for some constant  $\alpha > 0$ . Then the zero inertia equations (16) admit, for any smooth initial condition, a local smooth solution, which is global if  $d = 2$  and  $K = -\Delta$ . This solution can be obtained as the limit, as  $\epsilon$  goes to zero, of the weak solutions of the GNSB equations (10,11), with the same initial condition.*

For the convergence, we use a simple energy method. Namely, given a weak solution  $(y', v')$  to the GNSB equations (13,14,15) and a solution  $(y, v)$  of the HF equations, with same initial conditions  $(y^0, v^0)$ , we introduce

$$e(t) = \int_{T^d} (\epsilon |v'|^2 + |y - y'|^2) dx \quad (17)$$

and try to get an estimate of form:

$$\frac{d}{dt}(e(t) + O(\epsilon)) + \frac{1}{2} \int_{T^d} K(v - v') \cdot (v - v') dx \leq (e(t) + O(\epsilon))c, \quad (18)$$

where  $c$  depends only on the limit  $(y, v)$ , for any fixed finite time interval  $[0, T]$  on which  $(y, v)$  is smooth. From this estimate (18), we immediately get that  $y - y'$  and  $v - v'$  are of order  $O(\sqrt{\epsilon})$  in, respectively,  $L^\infty([0, T], L^2(T^d))$  and  $L^2([0, T], L^2(T^d))$ , using the coercivity of  $K$  ( $K \geq \alpha$  for some  $\alpha > 0$ ). So, we are left with proving (18). Notice first that, from equations (15) and (16), the following energy balances hold true:

$$\frac{1}{2} \frac{d}{dt} \int (\epsilon |v'|^2 + |y'|^2) dx + \int K v' \cdot v' dx \leq \int [F(x, y') \cdot v' + G(x, y') \cdot y'] dx$$

$$\frac{1}{2} \frac{d}{dt} \int |y|^2 dx + \int K v \cdot v dx = \int [F(x, y) \cdot v + G(x, y) \cdot y] dx.$$

Since  $(y', v')$  and  $(y, v)$  are respectively supposed to be a weak solution of the GNSB equations and a smooth solution of the zero inertia limit (16), we also get

$$-\frac{d}{dt} \int y \cdot y' dx$$

$$\begin{aligned}
 &= \int [((v \cdot \nabla)y) \cdot y' - ((v' \cdot \nabla)y) \cdot y'] dx - \int [G(x, y) \cdot y' + G(x, y') \cdot y] dx \\
 &= \int ((v \cdot \nabla)y - (v' \cdot \nabla)y) \cdot (y' - y) dx \\
 &\quad + \int [(G(x, y) - G(x, y')) \cdot (y - y') - G(x, y') \cdot y' - G(x, y) \cdot y] dx \\
 &\text{(using that } \int (w \cdot \nabla)y \cdot y \, dx = 0 \text{ for both } w = v \text{ and } w = v') \\
 &\leq c_1 \int (|v - v'| |y - y'| + |y - y'|^2) dx - \int [G(x, y') \cdot y' + G(x, y) \cdot y] dx
 \end{aligned}$$

where  $c_1$  depends on the Lipschitz constants of  $G$  (as a function of  $y$  and  $v$ ) and  $v$  (as a function of  $x$ ). Thus, adding up these three equalities, we get by definition (17):

$$\begin{aligned}
 &\frac{d}{dt} e(t) + \int (Kv' \cdot v' + Kv \cdot v) dx \\
 &\leq c_1 \int (|v - v'| |y - y'| + |y - y'|^2) dx + \int [F(x, y') \cdot v' + F(x, y) \cdot v] dx.
 \end{aligned}$$

From (13) and (16), we also get:

$$\int F(x, y) \cdot v' dx = \int (Kv + \nabla p) \cdot v' dx = \int Kv \cdot v' dx$$

and

$$\begin{aligned}
 \int F(x, y') \cdot v dx &= \epsilon \frac{d}{dt} \int v' \cdot v dx - \epsilon \int v' \cdot (\partial_t + v' \cdot \nabla) v dx + \int Kv' \cdot v dx \\
 &= \sqrt{\epsilon} \left( \frac{d}{dt} r_1(t) + r_2(t) \right) + e_2(t) + \int Kv' \cdot v dx,
 \end{aligned}$$

where

$$r_1(t) = \sqrt{\epsilon} \int v' \cdot v dx,$$

$$r_2(t) = -\sqrt{\epsilon} \int v' \cdot \partial_t v dx$$

and

$$e_2(t) = -\epsilon \int v' \cdot (v' \cdot \nabla) v dx.$$

Notice that  $r_1^2$ ,  $r_2^2$  and  $|e_2|$  are bounded by  $c_2 e(t)$  (by definition (17)), where  $c_2$  depends on the Lipschitz constant of  $v$  as a function of both  $t$  and  $x$ . So, we have obtained:

$$\frac{d}{dt} (e(t) - \sqrt{\epsilon} r_1(t)) + \int K(v' - v) \cdot (v' - v) dx$$

$$\leq c_1 \int (|v - v'| |y - y'| + |y - y'|^2) dx + \sqrt{\epsilon} r_2(t) + c_2 e(t).$$

Using the coercivity of  $K$  ( $K \geq \alpha > 0$ ) and definition (17), we find  $c_3$  depending on the Lipschitz constants of  $F$ ,  $G$  and  $v$  such that:

$$c_1 \int (|v - v'| |y - y'| + |y - y'|^2) dx \leq c_3 e(t) + \frac{1}{2} \int K(v' - v) \cdot (v' - v) dx.$$

This leads to:

$$\frac{d}{dt}(e(t) - \sqrt{\epsilon} r_1(t)) + \frac{1}{2} \int K(v' - v) \cdot (v' - v) dx \leq (c_2 + c_3) e(t) + \sqrt{\epsilon} r_2(t),$$

which leads to a differential inequality of the desired type, namely (18), since  $r_1^2 + r_2^2 \leq 2c_2 e$ . Thus, the proof of Theorem 2.2 is now achieved.

### 2.3 A Mechanical interpretation of the GNSB equations

In this subsection, we provide a mechanical interpretation of the GNSB equations and their zero inertia limit. For this purpose, it is worth considering the GNSB equations (10,11) in “Lagrangian coordinates”. Assuming the vector field  $v$  to be smooth enough, denoting by  $a \in D$  the position of a fluid parcel at time  $t = 0$ , we can recover its position  $X(t, a) \in D$  at later time  $t \geq 0$  by solving the ODE

$$\partial_t X(t, a) = v(t, X(t, a)), \quad X(0, a) = a, \quad \forall a \in D. \quad (19)$$

Notice that, for each  $t$ ,  $a \in D \rightarrow X(t, a) \in D$  is a measure preserving map as a consequence of the fact that  $v$  is a smooth divergence-free vector field parallel to  $\partial D$ . Let us also introduce:

$$Y(t, a) = y(t, X(t, a)) \in R^m. \quad (20)$$

Then, the GNSB equations (10,11) read in Lagrangian coordinates:

$$\epsilon \partial_{tt} X(t, a) + (Kv)(t, X(t, a)) + (\nabla p)(t, X(t, a)) = F(X(t, a), Y(t, a)), \quad (21)$$

$$\partial_t Y(t, a) = G(X(t, a), Y(t, a)),$$

where  $a \in D \rightarrow X(t, a) \in D$  is Lebesgue measure preserving. Let us now provide a possible mechanical interpretation. We model the atmosphere (or

the ocean) as a continuous distribution of infinitesimal rigid balloons floating inside  $D$ , each of them having position  $X(t, a)$  at time  $t$ , with  $X(0, a) = a$ , and being attached with probability  $\lambda(a) \geq 0$  to an anchor with an elastic cable. Of course, to be a realistic model with real balloons,  $\lambda$  should be a discrete probability distribution concentrated on a finite collection of points and the corresponding balloons should have a finite extension! Let us rather assume, for mathematical simplicity, that the balloons are just points and that  $\lambda$  is a smooth nonnegative density function on  $D$  with unit mass. The cable corresponding to the balloon labelled by  $a$  is modelled by a (possibly non Hookean) spring with restoring force  $-k(\xi, a) = k(-\xi, a) \in R^d$  where  $\xi \in R^d$  is the elongation of the spring. Notice that  $k$  may depend on  $a$ . The location of the anchor attached to the balloon labelled by  $a$  is not necessarily fixed and denoted by  $Y(t, a) \in R^d$ . (We may also think of an aircraft, or a boat, or any kind of carrier instead of an anchor.) Notice that we do not require the anchor to be located in  $D$ . Neglecting any interaction between the fluid and both the anchors and the springs (which may not be very realistic), we obtain the following dynamical equation for each balloon

$$\begin{aligned} \epsilon \partial_{tt} X(t, a) + (Kv)(t, X(t, a)) + (\nabla p)(t, X(t, a)) = \\ -\lambda(a)k(X(t, a) - Y(t, a), a). \end{aligned} \quad (22)$$

(Observe that as  $\lambda(a) = 0$ , the corresponding carrier  $Y(t, a)$  is just fictitious!) Let us consider the special case when the speed of each anchor is constant and given by:

$$\partial_t Y(t, a) = W(a) \in R^2. \quad (23)$$

Implicitly define a field  $y = (\tilde{y}, \hat{y}) = y(t, x) \in R^d \times D$  by setting

$$\tilde{y}(t, X(t, a)) = Y(t, a), \quad \hat{y}(t, X(t, a)) = a$$

(remember that  $a \rightarrow X(t, a)$  is supposed to be a diffeomorphism). Noticing that

$$\begin{aligned} ((\partial_t + v \cdot \nabla)y)(t, Y(t, a)) &= \partial_t[y(t, X(t, a))] = (\partial_t Y(t, a), 0) \\ &= (W(a), 0) = (W(\hat{y}(t, X(t, a))), 0) \end{aligned}$$

and going back to Eulerian coordinates, we recover the GNSB equations (10,11) in the particular case:

$$F(x, y) = -\lambda(\hat{y})k(x - \tilde{y}, \tilde{y}), \quad G(x, y) = (W(\hat{y}), 0). \quad (24)$$

(Notice that assuming  $\lambda$  and the restoring force  $k$  to have bounded derivatives up to order 2 is not very realistic! These assumptions are clearly made for mathematical convenience.) We may also consider the following variant of this mechanical model. Instead of prescribing their velocity by (23), we may assume that the carriers are driven by a friction-dominated retroaction of type:

$$\eta \partial_{tt} Y(t, a) + \partial_t Y(t, a) = -\mu(a)k(Y(t, a) - X(t, a), a),$$

where  $\mu$  is a given nonnegative function. Dropping the inertia term ( $\eta = 0$ ) leads to the following law:

$$\partial_t Y(t, a) = -\mu(a)k(Y(t, a) - X(t, a), a), \quad (25)$$

In that case, we get (keeping unchanged the definitions of  $Y$  and  $y$ ) again the GNSB equations (10,11) with  $F$  and  $G$  given by:

$$F(x, y) = -\lambda(\hat{y})k(x - \tilde{y}, \hat{y}), \quad G(x, y) = (-\mu(\hat{y})k(\tilde{y} - x, \hat{y}), 0). \quad (26)$$

Let us finally consider a second variant where the Coriolis force is added to the model (rotating ocean or atmosphere). Neglecting the vertical extension, so that  $d = 2$ , and assuming the rotation vector to be perpendicular to the ocean and of unit length, the Coriolis force  $Jv = (-v_2, v_1)$  is completely absorbed by the pressure term and the fluid parcels are not sensitive to it. (Indeed,  $v = (v_1, v_2)$  being divergence free,  $Jv$  is a gradient and can be removed from the dynamical equation.) However we may think that the carriers are still sensitive to the Coriolis force. Thus, we get for them

$$\eta \partial_{tt} Y(t, a) + J \partial_t Y(t, a) = -\mu(a)k(Y(t, a) - X(t, a), a),$$

instead of (25), neglecting a possible friction term. Neglecting the inertia term ( $\eta = 0$ ) leads to the balance equation:

$$\partial_t Y(t, a) = J\mu(a)k(Y(t, a) - X(t, a), a) \quad (27)$$

(using that  $J^2 = -J$ ). So we end up with a third version of the GNSB equations (10,11), for which:

$$F(x, y) = -\lambda(\hat{y})k(x - \tilde{y}, \hat{y}), \quad G(x, y) = (J\mu(\hat{y})k(\tilde{y} - x, \hat{y}), 0). \quad (28)$$

To summarize this subsection, let us just say that the GNSB equations (10,11) provide a rather flexible framework to describe the interaction between an incompressible fluid confined in a  $d$ -dimensional domain  $D$  (each



fluid parcel being labelled by its initial position  $a \in D$  having position  $X(t, a)$  at time  $t$ ), and a set of particles (also labelled by  $a$  and of position  $Y(t, a)$ ) moving in the ambient space  $R^d$ . The unusual feature of the resulting models is that the interaction is pairwise ( $X(t, a)$  interacts only with  $Y(t, a)$  for the same label  $a$ ), except for the mediation by the pressure field  $p(t, x)$  which preserves  $a \rightarrow X(t, a)$  in the class of Lebesgue measure preserving maps of  $D$  at each time  $t$ .

### 3 The Generalized Hydrostatic Boussinesq equations

In this final section, we investigate the most degenerate version of the GNSB equations (10,11), where we neglect not only the inertia terms but also the dissipative operator  $K$ . Thus we are left with the strange looking system:

$$F(x, y) = \nabla p, \quad \nabla \cdot v = 0, \quad (29)$$

$$\partial_t y + (v \cdot \nabla) y = G(x, y), \quad (30)$$

that we call Generalized Hydrostatic Boussinesq (GHB) equations (by seeing (29) as a generalization of the hydrostatic balance in Convection Theory). Let us concentrate on the simpler case when  $m = d$  and

$$F(x, y) = y - x \quad (31)$$

(which corresponds to cables modelled by Hookean springs, according to the mechanical interpretation of subsection 2.3). Thus, (29) just reads:

$$y = x + \nabla p(t, x). \quad (32)$$

In Lagrangian coordinates, the Generalized Hydrostatic Boussinesq equations (30,32) become:

$$Y(t, a) = X(t, a) + (\nabla p)(t, X(t, a)), \quad (33)$$

$$\partial_t Y(t, a) = G(X(t, a), Y(t, a)), \quad (34)$$

where, for all  $t$ ,  $a \in D \rightarrow X(t, a) \in D$  is a measure preserving map.

#### 3.1 Formal derivation of some optimal transport models from the GHB equations

We claim that several models involving optimal transport and the Monge-Ampère equation correspond to these GHB equations. In particular, we

consider the following generalization of the “semigeostrophic (SG) equations” [Ho, CNP, BB, CGP, Lo]:

$$\partial_t \rho + \nabla \cdot (\rho w) = 0, \quad (35)$$

$$w(t, x) = B \nabla \varphi, \quad (36)$$

$$\det(I + D^2 \varphi) = \rho, \quad (37)$$

where  $B$  is a  $d \times d$  constant matrix and  $D^2 \varphi$  is the “Hessian” matrix, made of all second order derivatives of  $\varphi(t, x)$  with respect to  $x$ . This system, that we call Generalized Semi-Geostrophic (GSG) equations involves the “fully nonlinear” Monge-Ampère (MA) equation (37) which, requires, in order to be of elliptic type, the convexity condition:

$$I + D^2 \varphi(t, x) > 0, \quad (38)$$

in the sense of symmetric matrices, for each time  $t$ . The 2D SG equations [Ho, CNP, BB, CGP, Lo] just correspond to the special case when  $d = 2$  and  $B$  is the rotation matrix of angle  $\pi/2$ . The case when  $B$  is just a number can be related to drift-diffusion and Keller-Segel type models [CMPS] for which the MA equation is replaced by the linear Poisson equation

$$\Delta \varphi = \rho - 1, \quad (39)$$

which can be seen as a linear approximation of the MA equation (37) as  $\varphi$  is small. The drift-diffusion case corresponds to (35,36,39) with  $B > 0$ . The simplified version of the Keller-Segel model [KS] treated by Jäger and Luckhaus [JL] corresponds to  $B < 0$  (with an additional diffusion term for  $\rho$  in equation (35)).

Let us now show that a solution of the GHB equations (33,34) corresponds to a solution of the GSG equations (35,36,37,38), in a suitable sense. For this purpose, in order to use the Polar Factorization Theorem 1.3, we make the following *a priori* assumptions for each time  $t$ :

A1: The map  $Y(t, \cdot)$  is non degenerate,

A2: The map  $x \in D \rightarrow x + \nabla p(t, x) \in R^d$  has a convex potential.

These assumptions mean that (33) defines the polar factorization of  $Y(t, \cdot)$  where  $X(t, \cdot)$  is measure preserving and  $x \in D \rightarrow x + \nabla p(t, x)$  has a convex

potential. According to the Polar Factorization Theorem 1.3, the measure preserving factor  $X(t, \cdot)$  can be written:

$$X(t, a) = (\nabla \Phi)(t, Y(t, a)),$$

where  $\Phi(t, x)$  is convex and Lipschitz continuous in  $x \in R^d$ , or, equivalently,

$$X(t, a) = Y(t, a) + (\nabla \varphi)(t, Y(t, a)), \quad (40)$$

where  $\varphi(t, x) = \Phi(t, x) - |x|^2/2$ . Since  $Y$  is supposed to be non degenerate (by assumption A1), there is a nonnegative Lebesgue integrable “density field”  $\rho(t, x)$  such that

$$\int_{R^d} f(x) \rho(t, x) dx = \int_D f(Y(t, a)) da, \quad (41)$$

for all suitable functions  $f$ . Thus:

$$\int_{R^d} f(x + \nabla \varphi(t, x)) \rho(t, x) dx = \int_{R^d} f(Y(t, a) + (\nabla \varphi)(t, Y(t, a))) da$$

(by definition (41) of  $\rho$ )

$$= \int_D f(X(t, a)) da = \int_D f(x) dx$$

(thanks to (40) and because  $X(t, \cdot)$  is Lebesgue measure preserving). So, we have obtained

$$\int_{R^d} f(x + \nabla \varphi(t, x)) \rho(t, x) dx = \int_D f(x) dx, \quad (42)$$

for all compactly supported continuous function  $f$ . This, combined with the assumption that  $x + \nabla p(t, x)$  has a convex potential can be seen as a weak form of the Monge-Ampère equation (37) combined with the ellipticity condition (38). Next, using (40), we can write (34) as

$$\partial_t Y(t, a) = w(t, Y(t, a)),$$

where  $w$  is the vector field defined by:

$$w(t, x) = G(x + \nabla \varphi(t, x), x), \quad (43)$$

which is nothing but a generalization of equation (36). Next, we get for all smooth compactly supported function  $f$  on  $R^d$ :

$$\frac{d}{dt} \int_D f(Y(t, a)) da = \int_D (\nabla f)(Y(t, a)) \cdot w(t, Y(t, a)) da,$$

which means, in terms of  $\rho$  defined by (41):

$$\frac{d}{dt} \int_{R^d} f(x) \rho(t, x) dx = \int_{R^d} \nabla f(x) \cdot w(t, x) \rho(t, x) dx,$$

that is (35) in a weak sense. So, we have fully recovered the GSG system (35,36,37,38), in a suitable weak form, from the GHB equations (33,34) under Assumption A1,A2. In addition, equation (36) can be replaced by the more general relation (43).

### 3.2 A global existence theorem for the GHB equations

We are now going to introduce a suitable concept of solutions of (33,34), by assuming *a priori* that, in (33), for each time  $t$ , the map  $x \in D \rightarrow x + \nabla p(t, x)$  has a convex potential and, therefore, is the unique rearrangement  $Y^*(t, \cdot)$  of  $Y(t, \cdot)$  in the class  $C$  of map with a convex potential, as in Theorem 1.2. (This corresponds to assumption A2 in the previous subsection.) Therefore, we can just write (33) as:

$$Y(t, a) = Y^*(t, X(t, a)).$$

From (34), we get (at least formally) that:

$$\frac{d}{dt} \int_D f(Y(t, a)) da = \int_D (\nabla f)(Y(t, a)) \cdot G(X(t, a), Y(t, a)) da.$$

for all compactly supported  $C^1$  function  $f$ . Thus,

$$\begin{aligned} \frac{d}{dt} \int_D f(Y^*(t, X(t, a))) da &= \\ &= \int_D (\nabla f)(Y^*(t, X(t, a))) \cdot G(X(t, a), Y^*(t, X(t, a))) da. \end{aligned}$$

Now, we can factor out  $X(t, a)$  and get a set of self-consistent equations for  $Y^*$ , namely:

$$\frac{d}{dt} \int_D f(Y^*(t, a)) da = \int_D (\nabla f)(Y^*(t, a)) \cdot G(a, Y^*(t, a)) da,$$

without loss of information for  $Y^*$ . This suggests the following concept of solution for the GHB equations (33,34):

**Definition 3.1** Assume that  $G$  is continuous and satisfies

$$\sup_{x,y} \frac{|G(x,y)|}{1+|y|} < \infty. \quad (44)$$

We say that  $Y^* \in C^0([0, T], L^2(D, R^d))$  is the “convex rearrangement” (CR) solution to the GHB equations (33,34), if:

- 1)  $Y^*(t, \cdot)$  belongs to the set  $C$  of all maps with convex potential, for all  $t \in [0, T]$ ,
- 2) For all compactly supported  $C^1$  function  $f$  on  $R^d$ , we have:

$$\frac{d}{dt} \int_D f(Y^*(t, a)) da = \int_D (\nabla f)(Y^*(t, a)) \cdot G(a, Y^*(t, a)) da. \quad (45)$$

This concept yields the following global existence theorem (without uniqueness) for all initial conditions in  $L^2$ :

**Theorem 3.2** For each initial condition  $Y^0 \in L^2(D, R^d)$ , there is at least one CR-solution  $Y^*(t, a)$ , in the sense of Definition 3.1, such that  $Y^*(t = 0, \cdot) = (Y^0)^*$ .

This solution can be obtained as the limit in  $C_t^0(L_a^2)$  as  $h \rightarrow 0$  of a time discrete approximation  $Y^h(t, a)$  defined, first at discrete times  $t = nh$ , by:

$$Y^h(nh + h, a) = [Y^h(nh, a) + h G(a, Y^h(nh, a))]^*, \quad n = 0, 1, 2, \dots \quad (46)$$

(where  $*$  denotes the rearrangement operator as in Theorem 1.2) and, then, linearly interpolated in  $t$ .

### Proof

To get the existence result, it is enough to show the convergence of the time discrete approximation  $Y^h$ . First, we observe that,  $Y^h(t, \cdot)$  is valued in  $C$ , (the class of maps with convex potential) for all time  $t$ . (This is true by definition for discrete times  $t = nh$  and preserved by linear interpolation since  $C$  is a convex cone.) Next, we deduce from (46) and assumption (44):

$$\sqrt{\int_D |Y^h(nh + h, a)|^2 da} \leq hc + (1 + hc) \sqrt{\int_D |Y^h(nh, a)|^2 da}, \quad (47)$$

for some constant  $c$  depending only on  $G$  and  $D$ . We also get, for all compactly supported  $C^1$  function  $f$ :

$$\begin{aligned} & \int_D [f(Y^h(nh + h, a)) - f(Y^h(nh, a))] da = \\ & = h \int_D \int_0^1 (\nabla f)(Y^h(nh, a) + h\theta G(a, Y^h(nh, a))) \cdot G(a, Y^h(nh, a)) d\theta da, \end{aligned} \quad (48)$$

which can be bounded by:

$$hc \sup_x \frac{|\nabla f(x)|}{1 + |x|} \int_D (1 + |Y^h(nh, a)|^2) da, \quad (49)$$

where  $c$  depends only on  $D$  and  $G$ .

So, from (47), we first see that  $Y^h(t, \cdot)$  is bounded in  $L^2$  uniformly in  $t \in [0, T]$  and  $h \in ]0, T]$  by some constant  $R$ . Therefore  $Y^h$  is uniformly valued in  $C_R$  the set of maps with convex potential and  $L^2$  norm bounded by  $R$ , which is a compact subset of  $L^2$ .

Next, we deduce from (48,49) that, for each fixed  $C^1$  function  $f$  such that

$$\frac{|\nabla f(x)|}{1 + |x|} < +\infty, \quad t \rightarrow \int_D f(Y^h(t, a)) da$$

is Lipschitz continuous on  $[0, T]$ , uniformly in  $h$ . Since Theorem 1.2 asserts the continuity of the map  $\mu \rightarrow y^*$ , we get that  $Y^h(t, \cdot)$  is uniformly equicontinuous from  $[0, T]$  to  $L^2$ . Then, we deduce from the Ascoli-Arzelà theorem that the set of all time-discrete approximations  $Y^h(t, a)$ , for  $0 < h \leq T$ , is relatively compact in  $C_t^0(L_a^2)$ . Thus, there is a sequence of time steps  $h$  for which  $Y^h$  converges to some limit  $Y^*$  in  $C_t^0(L_a^2)$ , which is necessarily valued in  $C_R$ , and therefore in  $C$ . We also easily get (45) by letting  $h$  go to zero in (48). So, the proof of Theorem 3.2 is now complete.

**Remark: continuous dependence as  $d = 1$**

In the very special case  $d = 1$ , the rearrangement operator  $*$  is well known to be non expansive in  $L^2$ :

$$\int_D |y^*(x) - z^*(x)|^2 dx \leq \int_D |y(x) - z(x)|^2 dx,$$

for pair  $(y, z)$  of  $L^2$  applications from  $D$  to  $R$ . It follows that two CR solutions  $y^*$  and  $z^*$ , obtained as limits of the time-discrete approximations (46), with respective initial condition  $y^0$  and  $z^0$ , must satisfy:

$$\int_D |y^*(t, x) - z^*(t, x)|^2 dx \leq \exp(ct) \int_D |y^0(x) - z^0(x)|^2 dx,$$

where  $c$  depends only on  $D$  and  $G$ .

## 4 Optimal transport and Magnetic Relaxation

In this section, we discuss a natural “stringy” generalization of the AHT equations (3,4) (discussed in the first part of the paper) and establish a link with the Arnold-Moffat model of Magnetic Relaxation (see [AK, Mo, Mo2, Sc, VMI, Ni]).

### 4.1 The AHT model as a gradient flow

Using Lagrangian coordinates, we deduce from (3,4), in the case when the dissipation operator is  $K = 1$ :

$$\partial_t X(t, a) + (\nabla p)(t, X(t, a)) = y^0(a) - X(t, a) \quad (50)$$

where  $X(t, \cdot)$  belongs to the set  $MPM(D)$  of all (Borel) Lebesgue measure preserving maps of  $D$ . (These equations can be either derived directly from (3,4) or obtained from the GNSB equations written in Lagrangian coordinates (21), by setting  $\epsilon = 0$ ,  $K = I$ ,  $F(x, y) = y - x$  and  $G(x, y) = 0$ .) As explained in [AHT], in slightly different words, the AHT equation (50) formally corresponds to the gradient flow of the “energy”

$$X \rightarrow \frac{1}{2} \int_D |X(a) - y^0(a)|^2 da, \quad (51)$$

on the “manifold” of all  $X \in MPM(D)$  for the  $L^2$  metrics. Let us recall that, as seen in the first section, minimizing this energy is equivalent to solve an Optimal Transport problem. The gradient flow structure can be easily understood by considering the standard time discretization of such a gradient flow. Let  $h > 0$  be a time step and let us denote by  $X^h(t, a)$  the discrete approximation of  $X(t, a)$  at discrete time  $t = nh$ ,  $n = 0, 1, 2, 3, \dots$ . At  $t = 0$ , we set  $X^h(0, a) = a$  and, for  $t = nh$ ,  $n = 1, 2, 3, \dots$ , we require  $X^h(t, \cdot)$  to be a minimizer among all  $X \in MPM(D)$  of the following functional:

$$X \rightarrow \int_D \frac{|X(a) - X^h(nh - h, a)|^2}{2h} + \frac{1}{2} \int_D |X(a) - y^0(a)|^2 da, \quad (52)$$

or, equivalently,

$$\int_D \left| X(a) - \frac{X^h(nh - h, a) + y^0(a)h}{1 + h} \right|^2 da \quad (53)$$

(after rearranging the squares).

Thus, assuming *a priori* that  $X^h(nh - h, \cdot) + y^0 h$  is non degenerate and using Theorem 1.3, this exactly means:

$$X^h(nh - h, \cdot) + y^0 h = y^* \circ X^h(nh, \cdot)$$

where  $y^*$  is a map with a convex potential. If we write this map as

$$x \rightarrow x + h(\nabla p^h)(nh, x),$$

we get

$$X^h(nh - h, \cdot) + y^0 h = X^h(nh, \cdot) + h(\nabla p^h)(nh, X^h(nh, \cdot)),$$

which can be seen just as a finite difference approximation of equation (50) as  $h \rightarrow 0$ . This is enough to interpret, at least formally, equation (50) as the  $L^2$  gradient flow of energy (51) on the “manifold”  $MPM(D)$ .

## 4.2 A stringy generalization of the AHT model

This analysis suggests the possibility of more complex models based on similar ideas. A rather natural idea amounts to consider, instead of the “manifold”  $MPM(D)$ , the “manifold” of “strings” valued in  $MPM(D)$ :  $s \in [0, 1] \rightarrow X(\cdot, s) \in MPM(D)$ , with fixed end values, say

$$X(t, a, s = 0) = X^-(t, a), \quad X(t, a, s = 1) = X^+(t, a). \quad (54)$$

Then, we may think of the  $L^2$  gradient flow of the following string energy:

$$\frac{1}{2} \int_0^1 \int_D |\partial_s X(a, s)|^2 da \, ds. \quad (55)$$

We claim that the resulting equation read:

$$\partial_t X(t, a, s) = \partial_{ss}^2 X(t, a, s) + (\nabla p)(t, X(t, a, s), s), \quad (56)$$

where  $X(t, \cdot, s)$  is valued in  $MPM(D)$  and end point conditions (54) are enforced. To get this system, as we did before for the AHT model, we define



a time discrete approximation  $X^h(t, a, s)$ , by setting  $X^h(0, a, s) = a$  and asking, for  $t = nh$ ,  $n = 1, 2, 3, \dots$ ,  $X^h(t, \cdot, \cdot)$  to be a minimizer among all curves  $s \in [0, 1] \rightarrow X(\cdot, s) \in MPM(D)$  of the functional:

$$\int_D \int_0^1 \frac{|X(a, s) - X^h(nh - h, a, s)|^2}{2h} da ds + \int_D \int_0^1 \frac{|\partial_s X(a, s)|^2}{2} da ds. \quad (57)$$

The formal optimality condition reads:

$$\begin{aligned} & \frac{X^h(nh, a, s) - X^h(nh - h, a, s)}{h} = \\ & = \partial_{ss}^2 X^h(nh, a, s) + (\nabla p^h)(nh, X^h(nh, a, s), s), \end{aligned}$$

for some scalar function  $p^h$ . So, we formally obtain, as  $h \rightarrow 0$ , the desired equation (56). Equation (56) has an interesting interpretation, obtained by assuming *a priori* that  $a \in D \rightarrow X(t, a, s)$  is a smooth orientation and measuring preserving diffeomorphism of  $D$  for each  $(t, s)$ . Then we introduce, for each  $(t, s)$ , two divergence free vector fields parallel to the boundary  $\partial D$ , namely  $v(t, x, s) \in R^d$  and  $b(t, x, s) \in R^d$ , defined by:

$$v(t, X(t, a, s), s) = \partial_t X(t, a, s), \quad b(t, X(t, a, s), s) = \partial_s X(t, a, s). \quad (58)$$

Then, we get from (56):

$$v = \partial_s b + (b \cdot \nabla)b + \nabla p, \quad (59)$$

while, from (58), we get the compatibility condition

$$\partial_t b + (v \cdot \nabla)b = (b \cdot \nabla)v + \partial_s v \quad (60)$$

(by writting  $\partial_{ts}^2 X = \partial_{st}^2 X$ ), to be added to the divergence free constraints

$$\nabla \cdot v = \nabla \cdot b = 0, \quad v // \partial D, \quad b // \partial D, \quad (61)$$

and the boundary conditions at  $s = 0$  and  $s = 1$  induced by (54), namely:

$$v(t, x, s = 0) = v^-(t, x), \quad v(t, x, s = 1) = v^+(t, x), \quad (62)$$

where  $v^+$  and  $v^-$  are prescribed. When the fields  $v$  and  $b$  do not depend on  $s$ , we get the Magnetic Relaxation model discussed by Moffatt in [Mo] (see also [AK, Mo2, Sc, VMI, Ni]). As  $t \rightarrow +\infty$ , we expect, at least for a large class

of initial conditions, the solution of equations (59,60,61) to converge toward an equilibrium, for which  $v = 0$  and  $b = b(x, s)$ ,  $p = p(x, s)$  are solutions to the Euler equations [AK, MP] ( $s$  acting as the time variable and  $b$  as the velocity field):

$$\partial_s b + (b \cdot \nabla) b + \nabla p = 0, \quad (63)$$

$$\nabla \cdot b = 0, \quad b \cdot \partial D. \quad (64)$$

Of course, we are far from being able to provide any rigorous proof of this conjecture.

### 4.3 The “cross-Burgers” equation

In the case when  $D$  is the unit ball, The Magnetic Relaxation equations (59,60,61) admit special solutions  $(b, v, \nabla p)$  which are linear in  $x$ :

$$b(t, x, s) = B(t, s)x, \quad v(t, x, s) = V(t, s)x, \quad \nabla p(t, x, s) = G(t, s)x, \quad (65)$$

where  $B, V$  are skew-symmetric matrices, while  $G$  is a symmetric matrix, all depending only on  $(t, s)$ . (Notice that the fields  $b$  and  $v$  are automatically parallel to the boundary  $\partial D$  since  $D$  is the unit ball.) The resulting equations for  $B, V$  and  $G$  are:

$$V = \partial_s B + B^2 + G, \quad (66)$$

$$\partial_t B + [V, B] = \partial_s V. \quad (67)$$

Since  $B^2$  is a symmetric matrix, equation (66) reduces to:

$$V = \partial_s B.$$

Thus, we get a single equation for  $B$ :

$$\partial_t B + [\partial_s B, B] = \partial_{ss}^2 B, \quad (68)$$

where  $[A, B]$  denotes the skew product  $AB - BA$ . In the special case  $d = 3$ ,  $B$  can be identified as a 3-vector and  $[\cdot, \cdot]$  as the cross product  $\times$  in  $R^3$ , which leads to:

$$\partial_t B + \partial_s B \times B = \partial_{ss}^2 B. \quad (69)$$

that we could call the “cross-Burgers” equation. This equation admits interesting special solutions, such as:

$$B(t, s) = (\alpha(t) \cos s, \alpha(t) \sin s, \beta(t) - 1)$$

where  $\alpha \geq 0$  and  $\beta$  are solutions to:

$$\frac{d\alpha}{dt} = -\beta\alpha, \quad \frac{d\beta}{dt} = \alpha^2,$$

or, equivalently,

$$\frac{d^2\lambda}{dt^2} + \exp(2\lambda) = 0,$$

where  $\lambda = \log(\alpha)$ .

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